ON G-FANO THREEFOLDS

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Abstract. We study Fano threefolds with terminal singularities admitting a "minimal" action of a finite group. We prove that under certain additional assumptions such a variety does not contain planes. We also obtain an upper bounds of the number of singular points of certain Fano threefolds with terminal factorial singularities.

1. Introduction

Let X be an algebraic variety over a field $\Bbbk$ of characteristic zero and let G be some group. We say that X is a G-*variety*, if the group G acts on $\overline{X} = X \otimes \bar{\Bbbk}$, where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$. Moreover, we assume that X, G and $\Bbbk$ satisfy one of the following conditions.

(a) *Geometric case*: the field $\Bbbk$ algebraically closed, the group G is finite and the action of G on X is defined by a homomorphism $G \to \operatorname{Aut}_{\Bbbk}(X)$.

(b) *Algebraic case*: G is the Galois group of $\bar{\Bbbk}$ over $\Bbbk$ acting on $\overline{X} = X \otimes \bar{\Bbbk}$ through the second factor. The action of G on X is trivial.

A G-variety X is called a G-*Fano variety*, if the singularities of X are not worse than terminal Gorenstein, the anticanonical divisor $-K_X$ is ample and the rank of the invariant part $\operatorname{Cl}(X)^G$ of the Weil divisor class group $\operatorname{Cl}(X)$ equals 1 (see [1]–[3]). In the present paper we consider only the three-dimensional case.

We say that a Fano threefold X *belongs to the main series*, if its canonical divisor K_X generates the Picard group $\operatorname{Pic}(X)$. G-Fano threefolds of non-main series were classified by the author in the works [2], [4].

Recall that *the genus* of a Fano threefold X is the number $\mathrm{g}(X) := \frac{1}{2}(-K_X)^3 + 1$ (see Definition 2.1).

We prove the following theorem.

Theorem 1.1. *Let X be a G-Fano threefold of the main series with $\mathrm{g}(X) \geq 6$. Then X does not contain any planes.*

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It turns out that the absence of planes on a Fano threefold with terminal singularities is very important for the classification. Indeed, for such a variety there exists a $\mathbb{Q}$-factorialization $\pi\colon X' \to X$, where X' is a variety with terminal factorial singularities and numerically effective (nef) big anticanonical divisor. If there are no planes on X, then running the minimal model program to X' we stay in the same class of varieties (the category of terminal factorial varieties with nef and big anticanonical divisor which does not contain planes; see [5], [6]). In some cases (especially for large values of genus) this allows to obtain a description of the original variety X. Applications of Theorem 1.1 that use this construction will be discussed in the forthcoming paper.

Note that for small values of genus, G-Fano threefolds can contain planes.

Example 1.2. *The Burkhardt quartic* X_4^{b} is the subvariety in $\mathbb{P}^5$, defined by the equations $\sigma_1 = \sigma_4 = 0$, where σ_i are elementary symmetric polynomials in $x_1, \ldots, x_6$. This quartic was intensively studied earlier (see, e.g., [7]). The singular locus of X_4^{b} consists of 45 ordinary double points. The symmetric group $\mathfrak{S}_6$ acts on X_4^{b} by permutations of coordinates. Then the quotient variety $\mathbb{P}^5/\mathfrak{S}_6$ is isomorphic to the weighted projective space $\mathbb{P}(1, \ldots, 6)$ and the quotient variety $X_4^{\mathrm{b}}/\mathfrak{S}_6$ is isomorphic to the subspace $\mathbb{P}(2,3,5,6) \subset \mathbb{P}(1, \ldots, 6)$. Therefore, $\operatorname{rk}\operatorname{Cl}(X_4^{\mathrm{b}})^{\mathfrak{S}_6} = 1$, and so X_4^{b} is a $\mathfrak{S}_6$-Fano threefold of the main series and genus 3. The quartic X_4^{b} contains exactly 40 planes [7].

It is known also a lot of examples of Fano threefolds of large genus (which are not G-Fano) with terminal singularities that contain planes. However the author does not know any examples of G-Fano threefolds of the main series of genus 4 and 5 containing planes (see Corollary 3.12).

$\mathbb{Q}$-factorial (terminal) Fano threefolds of the main series are always G-Fano with respect, for instance, to the trivial group. In this case, for $\mathrm{g}(X) \geq 8$, we obtain an upper bound for the number of singular points which is sharp for $\mathrm{g}(X) \geq 9$.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial Fano threefold of the main series with terminal singularities. Then, for* $\mathrm{g}(X) = 9, 10, 12$*, the number of singular points of X is at most* $12 - \mathrm{g}(X)$ *and this bound is sharp. For* $\mathrm{g}(X) = 8$ *the variety X has at most* 10 *singular points.*

Moreover, we generalize the classical Fano–Iskovskikh "double projection" construction (see Theorem 4.1).

The paper is organized as follows: § 2 is preliminary; in § 3 we prove Theorem 1.1; Theorem 1.3 is deduced from more general Theorem 4.1

in § 4; finally, § 5 contains two auxiliary results which are used in the proof Theorem 4.1.

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2. Preliminaries

For certainty, considering G-varieties, we deal only with the geometric case. The algebraic case is similar. Thus the ground field $\Bbbk$ is supposed to be algebraically closed (of characteristic 0). Sometimes we will assume also that $\Bbbk = \mathbb{C}$.

All the varieties considered in this paper have at worst terminal Gorenstein singularities. Every such a three-dimensional singularity $X \ni P$ is locally a hypersurface and has multiplicity 2. The Picard group of a variety with terminal singularities is embedded to the Weil divisor class group so that the cokernel has no torsion elements [8, Lemma 5.1].

Definition 2.1. Let X be a Fano threefold with terminal singularities. Its *genus* is the number $\mathrm{g}(X) := -K_X^3/2 + 1$.

By the Riemann–Roch formula and the Kawamata–Viehweg vanishing theorem we have $\dim|-K_X| = \mathrm{g}(X) + 1$. In particular, $\mathrm{g}(X)$ is an integer. In the case of Fano threefolds of the main series, the genus can take only the following values: $\mathrm{g}(X) \in \{2, 3, \dots, 10, 12\}$ (see [9] and [10]).

The Picard group $\mathrm{Pic}(X)$ and the Weil divisor class group $\mathrm{Cl}(X)$ are finitely generated and torsion free [9].

Theorem 2.2 ([11], [12])**.** *Let X be a Fano threefold with terminal singularities and with $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot K_X$. The following assertions hold:*

(i) *the linear system $|-K_X|$ is base point free;*

(ii) *if $\mathrm{g}(X) \geq 4$, then $|-K_X|$ is very ample and defines an embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$;*

(iii) *if $\mathrm{g}(X) \geq 5$, then the image $X = X_{2g-1} \subset \mathbb{P}^{g+1}$ is an intersection of quadrics.*

Theorem 2.3 ([10])**.** *Let X be a Fano threefold with terminal singularities. Then X is smoothable, i. e. there exists a flat family $\mathfrak{f} \colon \mathfrak{X} \to (\mathfrak{D} \ni 0)$ over a disk $(\mathfrak{D} \ni 0) \subset \mathbb{C}$ such that $\mathfrak{X}_0 \simeq X$ and a general element $\mathfrak{X}_s$, $s \in \mathfrak{D}$ is a nonsingular Fano threefold. Moreover, there exist natural identifications $\mathrm{Pic}(X) = \mathrm{Pic}(\mathfrak{X}_s) = \mathrm{Pic}(\mathfrak{X})$ so that $K_{\mathfrak{X}_s} = K_X$ (see [13, § 1]).*

Let (X, B) be a log pair (a pair consisting of a normal variety X and an effective $\mathbb{Q}$-divisor $B = \sum_i b_i B_i$ on X). Assume that $K_X + B$ is a $\mathbb{Q}$-Cartier divisor. Let $f \colon \tilde{X} \to X$ be a log resolution (X, B). Write

$$K_{\tilde{X}} = f^*(K_X + B) + E,$$

where $E = \sum_i e_i E_i$ is a $\mathbb{Q}$-divisor whose components are proper transforms of the components of B and the exceptional divisors. By our hypothesis $\sum_i E_i$ has only simple normal crossings. The pair (X, B) has *log canonical (lc) singularities* if $e_i \geq -1$ for all i. A proper irreducible subvariety $Z \subset X$ is called *a center of log canonical singularities* (X, B) if, for some resolution f, there exists a component E_i with coefficient $e_i \leq -1$ dominating Z. The union of all centers of log canonical singularities is called *the locus of log canonical singularities* and denoted by $\mathrm{LCS}(X, B)$. Thus,

$$\mathrm{LCS}(X, B) = \bigcup_{e_i \leq -1} f(E_i).$$

The sheaf

$$\mathcal{I}(X, B) := f_* \mathscr{O}_{\tilde{X}}(\lceil E \rceil)$$

is called *the multiplier sheaf*. Since B is effective, $\mathcal{I}(X, B)$ is an ideal sheaf. The corresponding subscheme in X is called *the scheme of log canonical singularities*. Its support coincides with $\mathrm{LCS}(X, B)$. If the pair (X, B) is lc, then $\mathscr{O}_X / \mathcal{I}(X, B)$ has no nilpotents and so the scheme of log canonical singularities is reduced (and coincides with $\mathrm{LCS}(X, B)$).

Nadel Vanishing Theorem (([14, Theorem 9.4.17])). *Let (X, B) be an lc pair, where the variety X is projective. Let D be a Cartier divisor on X such that the divisor $D - (K_X + B)$ is nef and big. Then*

$$H^q(\mathcal{I}(X, B) \otimes \mathscr{O}_X(D)) = 0 \quad \forall\, q > 0.$$

3. Planes

In this section we prove Theorem 1.1.

First we introduce the notation. Let X be a G-Fano threefold of the main series with terminal singularities. We assume that $g = \mathrm{g}(X) \geq 5$. Thus, $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot K_X$ and the linear system $|-K_X|$ defines an embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ so that its the image is an intersection of quadrics (by Theorem 2.2).

Assume that there exists a plane $\Pi_1 \subset X$. Let $O = \{\Pi_1, \ldots, \Pi_n\}$ be its orbit with respect to the action of G and let $D := \sum_i \Pi_i$. Recall that $\mathrm{Cl}(X)^G \simeq \mathrm{Pic}(X)^G \simeq \mathbb{Z} \cdot K_X$. Hence D is a Cartier divisor and

for some integer a we can write $D \sim -aK_X$. Comparing the degrees we obtain

$$n = (2g-2)a. \tag{3.1}$$

It is clear that for any two distinct planes $\Pi_i, \Pi_j \in O$ their intersection $\Pi_i \cap \Pi_j$ is either empty, a point or a line.

Lemma 3.1. *In the above notation, the number of planes passing through any point $P \in X \setminus \mathrm{Sing}(X)$ (and contained in X) is at most two. In particular, the divisor D has only simple normal crossings in the nonsingular locus $X \setminus \mathrm{Sing}(X)$.*

Proof. Let $P \in X$ be a nonsingular point and let $\Pi_1, \dots, \Pi_r \in O$ be all the planes passing through P. Then these planes are contained in the projective tangent space $\overline{T_{P,X}} \simeq \mathbb{P}^3$ to X at P. Since $X \subset \mathbb{P}^{g+1}$ is an intersection of quadrics (by Theorem 2.2), the subvariety $\overline{T_{P,X}} \cap X$ is an intersection of quadrics and so it cannot contain more than two planes. □

Corollary 3.2. *The pair (X, D) has only log canonical singularities in $X \setminus \mathrm{Sing}(X)$. Moreover, $X \setminus \mathrm{Sing}(X)$ does not contain any zero-dimensional log canonical centers.*

Lemma 3.3. *There are at most four planes passing trough a singular point $P \in X$ (and contained in X).*

Proof. As in Lemma 3.1, all the planes $\Pi_1, \dots, \Pi_r$ passing through P are contained in the set $\overline{T_{P,X}} \cap X$ which is an intersection of quadrics in $\overline{T_{P,X}}$. Since $P \in X$ is a hypersurface singularity, $\dim T_{P,X} = 4$. Since $\dim \overline{T_{P,X}} \cap X \le 2$, we we obtain that $\overline{T_{P,X}} \cap X$ contains at most four planes. □

Lemma 3.4. *The pair (X, D) is lc.*

Proof. Assume the contrary. Then $(X, (1-\varepsilon)D)$ is not lc for $0 < \varepsilon \ll 1$. According to Corollary 3.2 the locus of log canonical singularities $\mathrm{LCS}(X, (1-\varepsilon)D)$ is a finite set of points (non-empty), it is contained in the singular locus of X. On the other hand, by Shokurov's connectedness theorem (see [15, Theorem 17.4]) this set is connected[1]. Hence, $\mathrm{LCS}(X, (1-\varepsilon)D)$ is a single point P which must be G-invariant and singular for X. Then *all the components of* D pass through P. This contradicts Lemma 3.3 because the number of components of D greater than 4. The lemma is proved. □

[1] This argument works for $a = 1$. For $a > 1$ one can use the inversion of adjunction to a plane Π_i (see [16]) and the fact that $\overline{T_{P,X}} \cap X$ is an intersection of quadrics (see Theorem 2.2).

Since D is a Cartier divisor on a variety with terminal singularities, D is a Cohen–Macaulay scheme. Therefore, for any component $\Pi_i \subset D$, the intersection $\Pi_i \cap \operatorname{Supp}(D - \Pi_i)$ has pure dimension 1. On the other hand, the scheme $\Pi_i \cap \operatorname{Supp}(D - \Pi_i)$ is reduced in the generic point by Lemma 3.1 (and its components are distinct lines). Put $\Delta_i := \Pi_i \cap \operatorname{Supp}(D - \Pi_i)$.

Corollary 3.5. *For any component $\Pi_i \subset D$, the divisor $\Delta_i \subset \Pi_i$ has simple normal crossing.*

follows from Shokurov's log canonical inversion of adjunction (see [16]). $\square$

It is clear that two-dimensional centers of log canonical singularities of the pair (X, D) are planes Π_i and one-dimensional ones are those intersections $\Pi_i \cap \Pi_j$ that are lines. Denote by $\mathscr{P} \subset X$ the set of all zero-dimensional centers of log canonical singularities of (X, D). According to Corollary 3.2 we have $\mathscr{P} \subset \operatorname{Sing}(X)$.

Lemma 3.6. *For any component $\Pi_i \subset D$ we have $\mathscr{P} \cap \Pi_i = \operatorname{Sing}(\Delta_i)$.*

Proof. Fix a point $P \in \Pi_i$. Let $H \subset X$ be a general hyperplane section passing through P.

Let $P \in \operatorname{Sing}(\Delta_i)$. If P is not a log canonical center, then the pair $(X, D + \varepsilon H)$ is lc for $0 < \varepsilon \ll 1$. In this case, by the inversion of adjunction [16] the pair $(\Pi_i, \Delta_i + \varepsilon H|_{\Pi_i})$ is also lc which is impossible because the multiplicity of $\Delta_i + \varepsilon H|_{\Pi_i}$ at P is greater than 2. The contradiction shows that $\mathscr{P} \supset \operatorname{Sing}(\Delta_i)$.

Conversely, let $P \in \mathscr{P}$. Again by the inversion of adjunction the pair $(\Pi_i, \Delta_i + \varepsilon H|_{\Pi_i})$ is not lc for $0 < \varepsilon \ll 1$. Therefore, the curve Δ_i is singular at P. $\square$

Corollary 3.7. *For any point $P \in \mathscr{P}$, the divisor*

$$D^{(P)} := \sum_{\Pi_i \ni P} \Pi_i$$

is a cone with vertex P over a union of four lines forming a combinatorial cycle. In particular, the divisor $D^{(P)}$ has four components.

Proof. Let $H \subset X$ be a general hyperplane section. It is clear that $D^{(P)}$ is a cone over $H \cap D^{(P)}$ and $H \cap D^{(P)}$ is a union of lines. By Lemma 3.1 the divisor $H \cap D^{(P)}$ has simple normal crossing. If $H \cap D^{(P)}$ is not connected, then D can be decomposed in the sum $D' + D''$ of two effective divisors so that $D' \cap D'' = \{P\}$ (in a neighborhood of P). On the

other hand, since D is a Cartier divisor in a variety with terminal singularities, it is a Cohen–Macaulay scheme that leads to a contradiction. Thus, the intersection $H \cap D^{(P)}$ is connected.

Let $\Pi_i \subset D^{(P)}$. By Lemma 3.6 there exist exactly two components Δ_i passing through P. These components correspond to two planes Π_l, Π_k containing P. Therefore, each component $H \cap \Pi_i \subset H \cap D^{(P)}$ intersects exactly two other components $H \cap \Pi_l$ and $H \cap \Pi_k \subset H \cap D^{(P)}$. This means that $H \cap D^{(P)}$ is a combinatorial cycle.

Finally, the number of components of $H \cap D^{(P)}$ is at most 4 by Lemma 3.3 and this number cannot be less than 4, because H is an intersection of quadrics in $\mathbb{P}^g$ and so it does not contain "triangles" composed of lines. □

Lemma 3.8. *For each plane Π_i, the intersection $\Pi_i \cap \operatorname{Supp}(D - \Pi_i)$ has $2 + a$ one-dimensional components, where a is defined by the relation $D \sim -aK_X$ (cf. (3.1)).*

Proof. Let $H \subset X$ be a general hyperplane section. It is clear that H is a nonsingular K3 surface. Let $l_i := \Pi_i \cap H$. Since l_i is a nonsingular rational curve, we have

$$l_i \cdot \sum_{j \neq i} l_j = -l_i^2 + l_i \cdot \sum_j l_j = 2 + l_i \cdot \sum \Pi_i = 2 + a.$$

Thus Π_i intersects by lines exactly $2 + a$ components of D. □

Lemma 3.9. *We have $|\mathscr{P}| = (g-1)a(a+2)(a+1)/4$.*

Proof. Each plane $\Pi_i \in O$ contains $(a+2)(a+1)/2$ points from $\mathscr{P}$ (which form the whole singular locus of the union of $2 + a$ lines Δ_i) and there are exactly four planes $\Pi_j \in O$ passing through each point $P \in \mathscr{P}$. □

Lemma 3.10. *We have $|\mathscr{P}| = \dim |D|$.*

Proof. For $P \in \mathscr{P}$, let H_P be a general hyperplane section passing through P. Let $H := \sum_{P \in \mathscr{P}} H_P$ and let $B := (1-\delta)D + \varepsilon H$. Put $\mathcal{I}_{\mathscr{P}} := \mathcal{I}(X, B)$. For some $0 < \delta, \varepsilon \ll 1$, the pair (X, B) is lc and its locus of log canonical singularities $\operatorname{LCS}(X, B)$ coincides with $\mathscr{P}$. Since the pair (X, B) is lc, the scheme of log canonical singularities is reduced. Thus $\mathscr{O}_X/\mathcal{I}_{\mathscr{P}}$ is the structure sheaf of $\mathscr{P}$. Apply the Nadel Vanishing Theorem. We obtain $H^1(X, \mathcal{I}_{\mathscr{P}} \otimes \mathscr{O}_X(D)) = 0$. Then from the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathscr{P}} \otimes \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_{\mathscr{P}}(D) \longrightarrow 0$$

we obtain

$$|\mathscr{P}| = \dim H^0(\mathscr{O}_{\mathscr{P}}(D)) = \dim H^0(\mathscr{O}_X(D)) - \dim H^0(\mathcal{I}_{\mathscr{P}}(D)).$$

Since $\mathscr{P} \subset D$, we have $H^0(\mathcal{I}_{\mathscr{P}}(D)) \neq 0$. Therefore, $|\mathscr{P}| \leq \dim |D|$.

Assume that $|\mathscr{P}| \leq \dim |D| - 1$. Let $\mathscr{D} \subset |D|$ be the linear subsystem, consisting of divisors passing through all points of $\mathscr{P}$. Then $\dim \mathscr{D} \geq \dim |D| - |\mathscr{P}| \geq 1$. Assume that planes $\Pi_i, \Pi_j \in O$ intersect each other by a line l. Then $D \cdot l = a$. On the other hand, l contains exactly $a+1$ points from $\mathscr{P}$. Therefore any element $D' \in \mathscr{D}$ contains all the lines of the form $\Pi_i \cap \Pi_j$. In particular, $D' \cap \Pi_i$ contains $\Pi_i \cap \operatorname{Supp}(D - \Pi_i)$. Since the last set is a union of $a+2$ lines, we have $D' \supset \Pi_i$ for any i. Then $D' = D$, a contradiction. $\square$

Theorem 1.1. By the Riemann–Roch formula and Kawamata–Viehweg vanishing theorem we have

$$\dim |D| = \dim |-aK_X| = \frac{1}{12} a(a+1)(2a+1)(2g-2) + 2a.$$

Therefore, by Lemmas 3.9 and 3.10, we obtain

$$(a+1)(2a+1)(2g-2) + 24 = 3(g-1)(a+2)(a+1).$$

Thus we have

$$(a+1)(g-1)(4-a) = 24. \tag{3.2}$$

For $5 \leq g \leq 12$ the equation (3.2) has the following solutions:

$$(g, a, |\mathscr{P}|) = (5,1,6),\ (5,2,24),\ (7,3,90). \tag{3.3}$$

The last possibility is excluded by the lemma below. This proves our theorem. $\square$

Lemma 3.11 ([10])**.** *If $g \geq 6$, then $|\operatorname{Sing}(X)| \leq 29$.*

Proof. According to [10, Theorem 13] the number of singular points of a Fano threefold X with terminal singularities is at most

$$21 - \frac{1}{2}\operatorname{Eu}(\mathfrak{X}_s) = 21 - \frac{1}{2}\big(2 + 2\,\mathrm{b}_2(\mathfrak{X}_s) - \mathrm{b}_3(\mathfrak{X}_s)\big) = 20 - \rho(\mathfrak{X}_s) + \mathrm{h}^{1,2}(\mathfrak{X}_s),$$

where $\mathfrak{X}_s$ is a smoothing of X as in Theorem 2.3. In our case, $\rho(\mathfrak{X}_s) = \rho(X) = 1$ and $\mathrm{h}^{1,2}(\mathfrak{X}_s) \leq 10$ (see [9]). The lemma is proved. $\square$

For the case $\mathrm{g}(X) = 5$, from (3.3) we obtain the following partial result.

Corollary 3.12. *Let X be a G-Fano threefold of the main series with $\mathrm{g}(X) = 5$. Assume that X contains a plane Π_1 and let $\Pi_1, \dots, \Pi_n$ be its orbit. Let $\mathscr{P}$ be the set of all zero-dimensional log canonical centers of the pair $(X, \sum_i \Pi_i)$. Then has one of the following cases holds:*

(i) *$n = 8$, $|\mathscr{P}| = 6$, $|\operatorname{Sing}(X)| \geq 6$;*

(ii) *$n = 16$, $|\mathscr{P}| = 24$, $|\operatorname{Sing}(X)| \geq 24$.*

4. $\mathbb{Q}$-factorial case

In this section we generalize the Fano–Iskovskikh "double projection" method to the case of singular Fano threefolds.

Theorem 4.1. *Let X be a $\mathbb{Q}$-factorial Fano threefold of the main series with terminal singularities and $\mathrm{g}(X) \ge 7$. Then there exists the following diagram:*

$$
\begin{tikzcd}
 & Y \arrow[rr, dashed, "\chi"] \arrow[dl, "f"'] & & Y' \arrow[dr, "f'"] & \\
X & & & & Z
\end{tikzcd}
\tag{4.1}
$$

where f is the blowup of a line $l \subset X \setminus \mathrm{Sing}(X)$, χ is a flop, f' is a Mori contraction, and $\mathrm{Pic}(Z) \simeq \mathbb{Z}$.

(i) *If $g \ge 9$, then Z is a nonsingular Fano threefold and f' is the blow-up of an irreducible (possibly, singular) curve $B \subset Z$. Moreover, we have*

$\mathrm{g}(X)$	Z	$p_a(B)$	$-K_Z \cdot B$
9	$\mathbb{P}^3$	*3*	$4 \cdot 7$
10	$Q \subset \mathbb{P}^4$ – *nonsingular a quadric*	*2*	$3 \cdot 7$
12	$Z_5 \subset \mathbb{P}^6$ – *nonsingular del Pezzo threefold*	*0*	$2 \cdot 5$

$$|\,\mathrm{Sing}(X)| = |\,\mathrm{Sing}(B)| \le p_a(B).$$

In particular, X is nonsingular if $\mathrm{g}(X) = 12$.

(ii) *If $g = 8$, then f' is a conic bundle over $Z \simeq \mathbb{P}^2$ and the discriminant curve is (possibly, reducible) quintic $\Delta \subset \mathbb{P}^2$. Let r_1 be the number of ordinary double points Δ, r_2 be the number of simple cusps, and r_3 be the number of remaining singular points. Then*

$$|\,\mathrm{Sing}(X)| \le r_1 + r_2 + 2r_3.$$

(iii) *If $g = 7$, then f' is a del Pezzo fibration of degree 5 over $Z \simeq \mathbb{P}^1$.*

follows the classical idea of G. Fano (for a modern exposition for the nonsingular case we refer
Let X be a $\mathbb{Q}$-factorial Fano threefold of the main series with terminal singularities of genus $g = \mathrm{g}(X) \ge 7$. Then X is, in fact, factorial [8, Lemma 5.1] and the group $\mathrm{Cl}(X)$ is generated by the canonical class K_X. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{D} \ni o$ be a one-parameter smoothing of X as in Theorem 2.3. By the construction, a general fiber $X_s = \mathfrak{f}^{-1}(s)$ is a nonsingular Fano threefold and $\mathfrak{f}^{-1}(o) = X$. According to [18] each nonsingular fiber X_s contains a one-dimensional family of lines. Each line deforms to a one contained in $\mathfrak{X}$ and so the original variety X

also contains a one-dimensional family of lines $\mathscr{L}$. We claim that a general line l from this family $\mathscr{L}$ is contained in the nonsingular locus of X. Indeed, otherwise there exists a one-dimensional family of lines passing through one point $P \in X$ (because the singularities of X are isolated). All these lines swept out a surface $F \subset X \cap \overline{T_{P,X}}$ which must be a projective cone over some curve. Since X is an intersection of quadrics (by Theorem 2.2, (iii)) and $\dim \overline{T_{P,X}} = 4$, we have $\deg F \le 4$. On the other hand, $\mathrm{Cl}(X) = \mathbb{Z} \cdot K_X$, a contradiction.

Thus, $X \setminus \mathrm{Sing}(X)$ contains a line l. Hereinafter, the proof goes similar to that in [17]. However, since the threefold X can be singular in our case, some modifications are needed. For convenience of the reader we present the proof completely.

Let $f\colon Y \to X$ be the blowup of l, E be the exceptional divisor, and let $H := f^*(-K_X)$.

We need the following

Lemma 4.2 ((see, e.g., [9, Lemma 4.1.2] or (5.1))). *The following equalities hold:*
(4.2)
$$(-K_Y)^3 = 2g - 6, \quad (-K_Y)^2 \cdot E = 3, \quad (-K_Y) \cdot E^2 = -2, \quad E^3 = 1.$$

Using the same arguments as in [9, Sect.. 4.3.1] we show that the linear system $|-K_Y| = |H - E|$ is nef, big, and defines a birational morphism $\varphi\colon Y \to Y_{2g-6} \subset \mathbb{P}^{g-1}$.

It is easy also to show that $\dim |H - 2E| \ge g - 6$ (see, e.g., [17, § 2, Lemma 1]). Since $\mathrm{Cl}(X)$ is generated by the class of the divisor $-K_X$, for some $\alpha \ge 2$ the linear system $|H - \alpha E|$ has no fixed components. Using the relations (4.2), we obtain

$$\begin{aligned} 0 \le (-K_Y) \cdot (H - \alpha E)^2 &= (-K_Y) \cdot (-K_Y - (\alpha - 1)E)^2 \\ &= 2g - 6 - 6(\alpha - 1) - 2(\alpha - 1)^2. \end{aligned}$$

Since $g \le 12$, this gives us $\alpha = 2$, i. e. the linear system $|H - 2E|$ has no fixed components.

Further we claim that φ is a small morphism. Indeed, otherwise φ contracts a prime divisor D. For any fiber Υ of the morphism φ we have $-K_Y \cdot \Upsilon = 0$ and so $(H - 2E) \cdot \Upsilon < 0$. Therefore D is contained in the base locus of $|H - 2E|$. The contradiction shows that φ is a small crepant morphism.

In this situation there exists a flop $\chi\colon Y \dashrightarrow Y'$, where Y' has the same type of singularities as that of Y (terminal Gorenstein) [19]. Moreover, $\rho(Y') = \rho(Y) = 2$, the divisor $-K_{Y'}$ is nef, big, and the variety Y' (as Y) is factorial (see [8, Lemma 5.1]). Therefore, there

exists an extremal Mori contraction $f'\colon Y' \to Z$. According to the general theory of extremal rays there is the following exact sequence

$$0 \longrightarrow \operatorname{Pic}(Z) \xrightarrow{f'^*} \operatorname{Pic}(Y') \longrightarrow \mathbb{Z}, \tag{4.3}$$

where the map on the right hand side is defined by the intersection with some curve in a fiber. Hence, $\rho(Z) = \rho(Y') - 1 = 1$ and so $\dim Z > 0$. Let H' and E' be the proper transforms on Y' of the divisors H and E, respectively.

Lemma 4.3. *Let F be a divisor on Y' and D be its proper transform on Y. Write $D \sim \alpha(-K_Y) - \beta E$. Then*

$$\begin{aligned} (-K_Y)^2 \cdot D = (-K_{Y'})^2 \cdot F &= (2g-6)\alpha - 3\beta, \\ (-K_Y) \cdot D^2 = (-K_{Y'}) \cdot F^2 &= (2g-6)\alpha^2 - 6\alpha\beta - 2\beta^2. \end{aligned} \tag{4.4}$$

immediately follows from (4.2). □

Lemma 4.4. *Let D be a prime divisor on Y which is not big. For some integers α and β we write $D \sim \alpha(-K_Y) - \beta E$. Then*

$$\alpha, \beta > 0, \quad \beta \geq \alpha, \qquad (-K_Y)^2 \cdot D \geq 3\alpha + 2\beta.$$

Proof. Since $f(D)$ is effective, $\alpha > 0$. Since the divisor $-K_Y$ is big, $\beta > 0$. The divisors $-K_{Y'}$ and $-K_{Y'} - E'$ are nef on Y' and they are contained in the closed cone of ample divisors $\overline{\operatorname{Amp}}(Y')$. Hence D cannot be a convex linear combination of $-K_Y$ and $-K_Y - E$. Therefore, $\beta \geq \alpha$. Since the linear system $|-K_Y - E|$ has no fixed components, we have

$$(-K_Y) \cdot (-K_Y - E) \cdot D = (-K_Y)^2 \cdot D - (-K_Y) \cdot E \cdot D \geq 0.$$

On the other hand, $(-K_Y) \cdot E \cdot D = 3\alpha + 2\beta$ by (4.2). The lemma is proved. □

Below we consider the possibilities for the contraction f' according to the classification of extremal contractions [20].

Assume that $\dim Z = 1$. Then f' is a del Pezzo fibration and $Z \simeq \mathbb{P}^1$. Let F be a general geometric fiber. We use the notation of Lemma 4.3. Then $(-K_{Y'}) \cdot F^2 = 0$ and $(-K_{Y'})^2 \cdot F = K_F^2 \leq 9$. It follows from the sequence (4.3) that $\gcd(\alpha, \beta) = 1$ and it follows from the second relation in (4.4) that α divides 2. Note that $-\alpha K_F \sim \alpha(-K_{Y'})|_F \sim \beta E'|_F$. This means that the canonical divisor K_F of the del Pezzo surface F is divisible by β. Hence, $\beta \leq 3$. Moreover, if $\beta = 3$, then $F \simeq \mathbb{P}^2$ and $K_F^2 = 9$. In this case, taking (4.4) into account we obtain $(g-3)\alpha = 9$, $\alpha = 1$, and $(-K_Y) \cdot D^2 < 0$. On the other hand, $(-K_Y) \cdot D^2 = (-K_{Y'}) \cdot F^2 = 0$, a contradiction. Similarly, if $\beta = 2$,

then $K_F^2 = 8$, $(g-3)\alpha = 7$, $\alpha = 1$, and $(-K_Y)\cdot D^2 < 0$. Again we get a contradiction. Therefore, $\beta = \alpha = 1$ and again from (4.4) we obtain $g = 7$ and $K_F^2 = 5$, i. e. the case (iii) of our theorem.

Assume now that $\dim Z = 2$. According to [20] the surface Z is nonsingular and f' is a conic bundle. In our case, $\kappa(Z) = -\infty$ and $\rho(Z) = 1$. Hence, $Z \simeq \mathbb{P}^2$. Let $\Delta \subset \mathbb{P}^2$ be the discriminant curve, let $l \subset \mathbb{P}^2$ be a line, and let $F := {f'}^{-1}(l)$. Again we use the notation of Lemma 4.3. Since a general geometric fiber $C \subset Y'$ is a conic, we have $(-K_Y)\cdot C = (-K_Y)\cdot D^2 = 2$ and $0 = F\cdot C = 2\alpha - (E'\cdot C)\beta$. It follows from the sequence (4.3) that $\gcd(\alpha, \beta) = 1$ and so β divides 2. By Lemma 4.4 we have $\alpha = 1$. Since $(-K_Y)\cdot D^2 = 2$, the second relation in (4.4) has the form $2g - 6 - 6\beta - 2\beta^2 = 2$. Hence, $\beta = 1$ and $g = 8$. Finally, by the adjunction formula

$$K_F = (K_{Y'} + F)|_F, \qquad K_F^2 = K_{Y'}^2\cdot F + 2K_{Y'}\cdot F^2 = 3.$$

Therefore, the projection $f'|_F \colon F \to l$ has five degenerate fibers. Thus, $\deg\Delta = 5$. We obtain the case (ii) of our theorem.

Assume that the morphism f' is birational and contracts an (irreducible) divisor F to a point. Let, as above, $D \subset Y$ be the proper transform F and $D \sim \alpha(-K_Y) - \beta E$. According to the classification from [20] there exist four types of such contractions and in all these cases $(-K_{Y'})^2\cdot D' \le 4$. This contradicts Lemma 4.4.

Finally, assume that the morphism f' is birational and contracts an (irreducible) divisor F to a curve B. According to [20] the singularities of the curve B are locally planar, the variety Z is nonsingular along B, and f' is the blowup of the ideal sheaf of B. Then Z is a Fano threefold with terminal factorial singularities and $\rho(Z) = 1$. Let A be the positive generator of the group $\mathrm{Pic}(Z)$. Then $-K_Z = \iota A$ for some positive integer ι which is called the *Fano index*. It is well-known that $\iota \le 4$ (see [9] and Theorem 2.3). Moreover, $\iota = 4$ if and only if $Z \simeq \mathbb{P}^3$, and $\iota = 3$ if and only if Z is a quadric in $\mathbb{P}^4$.

Below we use the notation of Lemma 4.3. Let C be a general fiber of $f'|_F \colon F \to B$. Since over a general point of the curve B the morphism f' is a usual blowup, $F\cdot C = -1$. Therefore, $(E'\cdot C)\beta = \alpha + 1$. In particular, β divides $\alpha+1$. Since $\dim|F| = 0$ and $\dim|-K_{Y'} - E'| > 0$, we have $\alpha \ne \beta$. Then from Lemma 4.4 we obtain $\beta > \alpha$. Hence, $\beta = \alpha + 1$. Further,

$$K_{Y'} = (f')^*K_Z + F = -\iota(f')^*A + \alpha(-K_{Y'}) - (\alpha+1)E',$$
$$\iota {f'}^*A = (\alpha+1)(-K_{Y'} - E').$$

Since the divisors $(f')^*A$ and $-K_{Y'} - E$ are primitive elements of the lattice $\mathrm{Pic}(Y')$, we have $\beta = \alpha + 1 = \iota$ and $(f')^*A = -K_{Y'} - E'$. In

particular, $1 \le \alpha \le \iota - 1 = 3$. Moreover,

$$\dim |A| \ge |-K_Y - E| \ge g - 6. \tag{4.5}$$

The intersection theory on Y' has the same form as the intersection theory on the blowup of a nonsingular variety along a nonsingular curve (see (5.1)). Hence,

$$(-K_Y)^2 \cdot D = -K_Z \cdot B - 2p_a(B) + 2,$$
$$(-K_Y) \cdot D^2 = 2p_a(B) - 2.$$

Taking (4.4) and $\beta = \alpha + 1$ into account we obtain

$$(2g-6)\alpha - 3(\alpha+1) = -K_Z \cdot B - 2p_a(B) + 2,$$
$$(2g-6)\alpha^2 - 6\alpha(\alpha+1) - 2(\alpha+1)^2 = 2p_a(B) - 2.$$

Adding up the last two equalities we obtain

$$(2g-6)\alpha - 3(\alpha+1) + (2g-6)\alpha^2 - 6\alpha(\alpha+1) - 2(\alpha+1)^2 = -K_Z \cdot B.$$

Since $\beta = \alpha + 1 = \iota$, we have

$$2(g-7)\alpha = A \cdot B + 5.$$

Consider cases $\alpha = 1, 2, 3$ separately.

Let $\alpha = 1$. Then $\beta = \iota = 2$ and

$$(2g-6)\alpha^2 - 6\alpha\beta - 2\beta^2 = 2g - 26 = 2p_a(B) - 2. \tag{4.6}$$

The only solution for (4.6) is $p_a(B) = 0$, $g = 12$, $A \cdot B = 5$. In this case, Z is a del Pezzo threefold (see, e.g., [9] or [2]). According to (4.5) we have $\dim |A| \ge 6$. Since $\rho(Z) = 1$ and Z factorial, $A^3 = 5$. According to [2, Corollary 5.4] the variety Z is nonsingular.

Let $\alpha = 2$. Then $\beta = \iota = 3$ and Z is a quadric in $\mathbb{P}^4$. Since the variety Z is factorial, this quadric is nonsingular. As above, we have

$$(2g-6)\alpha^2 - 6\alpha\beta - 2\beta^2 = 8g - 78 = 2p_a(B) - 2, \quad 4g = p_a(B) + 38. \tag{4.7}$$

According to (4.5) we have $4 = \dim |A| \ge g - 6$. Hence, $g \le 10$. The only solution of (4.7) is $g = 10$, $p_a(B) = 2$, $A \cdot B = 7$.

Let $\alpha = 3$. Then $\beta = \iota = 3$ and $Z \simeq \mathbb{P}^3$. As above, $9g = p_a(B) + 78$, $3 = \dim |A| \ge g - 6$, $g = 9$, $p_a(B) = 3$, $A \cdot B = 7$.

Thus the existence of the diagram (4.1) and its properties are proved.

For the proofs of the assertion about singularities we have to notice that, by the construction, f is an isomorphism near $\operatorname{Sing}(X)$ and $\operatorname{Sing}(Y)$, and the map χ preserves completely the type of singularities (and their number) [19]. Thus, $|\operatorname{Sing}(X)| = |\operatorname{Sing}(Y')|$. The bound for $|\operatorname{Sing}(Y')|$ follows from Proposition 5.2 in the case $g = 8$ and from Proposition 5.1 in cases $g \ge 9$. The theorem is proved. $\square$

Remark 4.5. For $g \geq 9$, the construction (4.1) can be reversed: for a suitable choice of the curve B with corresponding values of degree and arithmetic genus its the blowup $f'\colon Y' \to Z$ satisfies the standard conditions:

a) the linear system $|-K'_Y|$ is base point free;

b) the corresponding morphism $\Phi_{|-K'_Y|}$ does not contract any divisors;

c) the variety Y' has only terminal singularities.

In this situation, there exists (and reconstructed uniquely) the right hand part of the diagram (4.1). Such a curve can be chosen on a nonsingular del Pezzo surface of degree 3, 4, 5 in cases $g = 9, 10, 12$, respectively. This allows to resolve problems on the existence of Fano threefolds with given number of singular points.

In particular, this construction allows to construct examples of nonprojective Moishezon threefolds with $b_2 = 1$ as small resolutions of our variety X (cf. [21]).

Theorem 1.3. In the cases $g \geq 9$ the assertion immediately follows from Theorem 4.1, (i). Consider the case $g = 8$. For a plane reduced (but possibly reducible) curve C put $\gamma(C) := r_1 + r_2 + 2r$, where r_1 (respectively, r_2) is the number of simple double points of type A_1 (respectively, the number of double points of type A_2), and r is the number of remaining singular points. Then by Theorem 4.1, (ii) we have $|\operatorname{Sing}(X)| \leq \gamma(\Delta)$. The estimate $\gamma(\Delta) \leq 10$ for a plane quintic Δ follows from the following two simple assertions:

1) if the curve C is irreducible, then $\gamma(C) \leq p_a(C)$;

2) if C_1 is an irreducible nonsingular component of C, then $\gamma(C) \leq \gamma(C - C_1) + C \cdot (C - C_1)$.

The theorem is proved. $\square$

Remark 4.6. In contrast with the case $g \geq 9$, we do not assert that the bound $|\operatorname{Sing}(X)| \leq 10$ is sharp for $g = 8$. One can conjecture that it can be improved.

5. Two auxiliary results

Proposition 5.1. *Let V be a threefold with terminal singularities and let $f\colon V \to W$ be a birational Mori contraction that contracts a divisor F to a curve B. Then the following assertions hold:*

(i) *the singularities of the curve B are locally planar, the variety W is nonsingular along B, and f is the blowup of the ideal sheaf of B;*

(ii) *$f(\operatorname{Sing}(X) \cap F) = \operatorname{Sing}(B)$ and each fiber $f^{-1}(b)$ over a point $b \in \operatorname{Sing}(B)$ contains exactly one singularity of X.*

Moreover,

$$(5.1)\qquad \begin{gathered}(-K_V)^3 = (-K_W)^3 + 2K_W \cdot B + 2p_a(B) - 2,\\ (-K_V)^2 \cdot E = -K_W \cdot B - 2p_a(B) + 2,\\ (-K_V) \cdot E^2 = 2p_a(B) - 2.\end{gathered}$$

Proof. The assertion (i) follows from [20] and the assertion (ii) is a simple computation in local coordinates. Let us prove (5.1). For some ample divisor A on W, the linear system $|-K_V + f^*A|$ has no base points (see [20, Proposition 1]). Take a general element $S \in |-K_V + f^*A|$. By Bertini's theorem S is nonsingular. Let $\bar{S} := f(S)$. Then $\bar{S} \in |-K_W + A|$ and $f^*\bar{S} = S + E$. The restricted linear system $|-K_V+f^*A|\big|_E$ is ample and base point free. Again by Bertini's theorem its general element $S \cap E$ is a nonsingular irreducible curve. Since the intersection number of S and a general fiber $E \to f(E)$ equals 1, the restriction $f_S\colon S \to \bar{S}$ is an isomorphism and $f_S(E \cap S) = B$. Note that $K_S = f^*A|_S$, $K_{\bar{S}} = A|_{\bar{S}}$, and $(B \cdot B)_{\bar{S}} = 2p_a(B) - 2 - A \cdot B$. Using the last relation we can write

$$\begin{gathered}-K_V \cdot E^2 = (S - f^*A) \cdot E^2 = (B \cdot B)_{\bar{S}} + A \cdot B = 2p_a(B) - 2,\\ -K_V \cdot f^*K_W \cdot E = (S - f^*A) \cdot f^*K_W \cdot E = S \cdot f^*K_W \cdot E = K_W \cdot B.\end{gathered}$$

Hence we have

$$\begin{aligned}K_V^2 \cdot E &= K_V \cdot f^*K_W \cdot E + K_V \cdot E^2 = -K_W \cdot B - 2p_a(B) + 2,\\ (-K_V)^3 &= -K_V \cdot (f^*K_W + E)^2 = -(f^*K_W + E) \cdot f^*K_W^2\\ &\quad - 2K_V \cdot f^*K_W \cdot E - K_V \cdot E^2 = (-K_W)^3 + 2K_W \cdot B + 2p_a(B) - 2.\end{aligned}$$

The proposition is proved. □

Proposition 5.2. *Let V be a threefold with terminal singularities and let $f\colon V \to W$ be a Mori contraction to a surface. Then the surface W is nonsingular and f is a conic bundle (possibly, singular). Further, let $\Delta \subset W$ be the discriminant curve. Then $f(\mathrm{Sing}(V)) \subset \mathrm{Sing}(\Delta)$. Moreover, any fiber $f^{-1}(w)$, $w \in \mathrm{Sing}(\Delta)$ contains at most two points of $\mathrm{Sing}(V)$. If $f^{-1}(w) \cap \mathrm{Sing}(V)$ consists of exactly two points, then the singularity $w \in \Delta$ is not an ordinary double point A_1 nor a simple cusp A_2.*

Proof. The first part of the proposition is contained in [20]. It remains to prove only the assertion about singularities of V. Since the problem is local, we may assume that the ground field $\Bbbk$ is the field of complex numbers $\mathbb{C}$, V is an analytic neighborhood of a fiber $f^{-1}(w)$, and $W \subset \mathbb{C}^2_{u,v}$ is a small disk containing $w = (0,0)$.

Then we can embed V to $\mathbb{P}^2 \times W$ so that V is defined by the equation $q(x, y, z; u, v) = 0$, where q is regarded as a quadratic form in x, y, z with coefficients in $\mathbb{C}\{u, v\}$. The fiber $f^{-1}(w)$ is defined by the equation $q(x, y, z; 0, 0) = 0$. Since $f^{-1}(w)$ is a conic, we have $\operatorname{rk} q(x, y, z; 0, 0) \geq 1$. If $\operatorname{rk} q(x, y, z; 0, 0) = 3$, then the fiber $f^{-1}(w)$ is nonsingular and V is also nonsingular (near $f^{-1}(w)$). If $\operatorname{rk} q(x, y, z; 0, 0) = 2$, then up to coordinate change we can write $q(x, y, z; 0, 0) = x^2 + y^2$ and $q(x, y, z; u, v) = x^2 + y^2 + \alpha(u, v)z^2$, where $\alpha = 0$ is the equation of Δ and $\alpha(0, 0) = 0$. In this case, V is singular, if and only if $\operatorname{mult}_{(0,0)} \alpha > 1$, i. e. the curve Δ is singular at the origin. Moreover, $\operatorname{Sing}(V) \subset \operatorname{Sing}(f^{-1}(w)) = \{P\}$, where $\operatorname{Sing}(f^{-1}(w))$ is a single point.

Finally, consider the case $\operatorname{rk} q(x, y, z; 0, 0) = 1$. Then up to coordinate change we can write $q(x, y, z; 0, 0) = x^2$ and $q(x, y, z; u, v) = x^2 + \alpha y^2 + 2\beta yz + \gamma z^2$, where α, β, γ are holomorphic functions in u, v, vanishing at the origin. The equation of Δ has the form $\alpha\gamma - \beta^2 = 0$. Hence the curve Δ is singular at $(0, 0)$. Assume that V has two singular points P_1, P_2 on $f^{-1}(w)$. By changing the coordinates y, z linearly, we may assume that $P_1 = (0 : 1 : 0)$, $P_2 = (0 : 0 : 1)$. Then $\operatorname{mult}_{(0,0)} \alpha > 1$ and $\operatorname{mult}_{(0,0)} \gamma > 1$. Since the singularities of V are isolated, we have $\operatorname{mult}_{(0,0)} \beta = 1$. Then it is easy to see that the variety V is nonsingular outside of P_1, P_2 and the singularity $\{\alpha\gamma - \beta^2 = 0\}$ is not an ordinary double point nor a simple cusp. $\square$

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